Lax pair tensors in arbitrary dimensions

1999 J. Phys. A: Math. Gen. 32 3377

(http://iopscience.iop.org/0305-4470/32/18/311)

J. Phys. A: Math. Gen. **32** (1999) 3377–3383. Printed in the UK PII: S0305-4470(99)98558-X

Lax pair tensors in arbitrary dimensions

Martin Goliath†, Max Karlovini‡ and Kjell Rosquist§
Department of Physics, Stockholm University, Box 6730, S-11385 Stockholm, Sweden

Received 21 October 1998, in final form 18 February 1999

Abstract. A recipe is presented for obtaining Lax tensors for any n-dimensional Hamiltonian system admitting a Lax representation of dimension n. Our approach is to use the Jacobi geometry and coupling-constant metamorphosis to obtain a geometric Lax formulation. We also exploit the results to construct integrable spacetimes, satisfying the weak energy condition.

1. Introduction

In this paper we extend the geometric formulation of the Lax pair equation given in [1, 2]. In [2] a canonical transformation was used to formulate the three-particle non-periodic Toda system as geodesic equations of a three-dimensional Riemannian space. However, the canonical transformation which was used depends on the particular system and so the method has no obvious extension to more general situations. A standard way to geometrize a system is to reparametrize the time variable leading to the Jacobi geometry, see e.g. [3]. This approach has the advantage that it always works for Hamiltonians of the type $H = T + V$ with a quadratic kinetic energy. However, it was not understood how to transform the Lax representation to the Jacobi time gauge. To remedy this situation we give a recipe for transforming any Lax representation to the Jacobi time gauge by using the method of coupling-constant metamorphosis [4]. It turns out that the resulting Lax system is again homogeneous of degree one in the momenta. However, unlike the previous examples it has a nonlinear dependence on the momenta with some terms being proportional to the square root of the Jacobi Hamiltonian. As a result the original geometric formulation of the Lax pair equation [1] cannot be used as it stands in this context. Instead, one is led to a slightly more general geometric Lax representation.

The approach to geometric formulation of integrable systems presented in this paper has both advantages and drawbacks. The advantages are that the dimension of the system is unchanged, that the metric is just a conformal rescaling of the original kinetic metric (usually flat space) and that the method works for any integrable system with quadratic kinetic energy and with an arbitrary number n, of particles. A requirement for the method to work is that the corresponding Lax representation is of dimension n. It is a drawback that the time must be reparametrized and also that the geometric Lax formulation involves two dynamical tensors instead of one as was the case in the original formulation. In any case, we consider this work as a further step towards a more complete understanding of integrable geometries and their associated Lax systems.

† E-mail address: goliath@physto.se
‡ E-mail address: max@physto.se
§ E-mail address: kr@physto.se

We also apply our results to construct two types of integrable spacetimes satisfying the weak energy condition in an open region.

2. Transforming Lax pair representations to Jacobi time

A common feature of completely integrable Hamiltonian systems is the existence of a Lax pair, i.e. a pair of matrices $(\boldsymbol{L}, \boldsymbol{A})$ satisfying the equations of motion

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = [\boldsymbol{L}, \boldsymbol{A}] \tag{1}$$

where the time derivative is defined by

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \{\boldsymbol{L}, H\}. \tag{2}$$

It follows that $I_k = k^{-1}\,\mathrm{Tr}\,\boldsymbol{L}^k$, $k = 1, 2, \ldots$, is a sequence of invariants of the system. We consider n-dimensional Hamiltonians of the classical type

$$H = T + V \qquad T = \tfrac{1}{2}h^{\alpha\beta}p_\alpha p_\beta \qquad V = V(q) \tag{3}$$

where $h_{\alpha\beta}$ is the kinetic metric. Furthermore, we restrict to Lax matrices of dimension n that are linear (but not necessarily homogeneous) in the momenta:

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{L}^\alpha p_\alpha + \boldsymbol{K} \\ \boldsymbol{A} &= \boldsymbol{A}^\alpha p_\alpha + \boldsymbol{D}. \end{aligned} \tag{4}$$

Here, $\boldsymbol{L}^\alpha$, $\boldsymbol{A}^\alpha$, $\boldsymbol{K}$ and $\boldsymbol{D}$ are independent of the momenta. We seek a general recipe for transforming such a Lax representation under the Jacobi time transformation $t \to t_J$, $\mathrm{d}t_J = 2(E - V)\mathrm{d}t$, which maps orbits of an energy surface $H = E$ into geodesics of the Jacobi geometry $g_{\alpha\beta} = 2(E - V)h_{\alpha\beta}$, i.e. into orbits of the Jacobi Hamiltonian $H_J = \frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta = [2(E - V)]^{-1}T$. This is naturally accomplished by performing the time transformation as a coupling-constant metamorphosis [4]. To this end, we first of all need to introduce a coupling constant into our original Hamiltonian. This is accomplished by making the rescalings $p_\alpha \to \lambda^{-1}p_\alpha$ and $H \to \lambda^{-2}H$, resulting in

$$H = T + \lambda^2 V \tag{5}$$

where T is defined by its original functional form. The transformation is noncanonical, yet when accompanied by the time rescaling $t \to \lambda t$, the canonical equations of motion are preserved. However, the same does not hold true for the Lax equation (1), but this is a drawback which can be just as easily cured by making another rescaling, namely $\boldsymbol{A} \to \lambda^{-1}\boldsymbol{A}$. Finally, we fix the gauge freedom to make arbitrary rescalings of $\boldsymbol{L}$ by letting $\boldsymbol{L} \to \lambda^{-1}\boldsymbol{L}$, which gives the rescaled Lax pair

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{L}^\alpha p_\alpha + \lambda\boldsymbol{K} \\ \boldsymbol{A} &= \boldsymbol{A}^\alpha p_\alpha + \lambda\boldsymbol{D}. \end{aligned} \tag{6}$$

The passage to the Jacobi time t_J, can now be obtained as a coupling-constant metamorphosis acting on the coupling constant $\kappa := \frac{1}{2}\lambda^2$ which has entered the Hamiltonian. However, to end up with the homogeneous Hamiltonian H_J when solving a fixed energy constraint for κ, we need to hold on to the interpretation of the parameter E as the energy value of the original Hamiltonian $H = T + V$, thereby making $\lambda^2 E$ the corresponding energy value of the rescaled Hamiltonian $H = T + \lambda^2 V$. This, in fact, means that we are not really dealing with a coupling-constant metamorphosis in the original sense, as the old energy E does not enter linearly into the new Hamiltonian H_J. Nevertheless, it is not difficult to realize that the results of [4] still apply. It thus follows that when substituting H_J for κ (i.e. $\sqrt{2H_J}$ for λ) in the expressions for H, $\boldsymbol{L}$ and $\boldsymbol{A}$ above, the original Lax equation (1) can be written as

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t_J} = [2(E-V)]^{-1}[\boldsymbol{L}, \boldsymbol{A}] \tag{7}$$

where

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t_J} = \{\boldsymbol{L}, H_J\} \tag{8}$$

and the factor $[2(E-V)]^{-1}$ in front of the matrix commutator arises from the fact that $\mathrm{d}/\mathrm{d}t = 2(E-V)\,\mathrm{d}/\mathrm{d}t_J$. Absorbing this factor into $\boldsymbol{A}$, we obtain the final time transformed Lax equation. To summarize, the original Lax pair system with H, $\boldsymbol{L}$ and $\boldsymbol{A}$ given by equations (3) and (4), is transformed to the Jacobi time gauge according to

$$\begin{aligned} H_J &= [2(E-V)]^{-1}T \\ \boldsymbol{L}_J &= \boldsymbol{L}^\alpha p_\alpha + \sqrt{2H_J}\boldsymbol{K} \\ \boldsymbol{A}_J &= [2(E-V)]^{-1}\left(\boldsymbol{A}^\alpha p_\alpha + \sqrt{2H_J}\boldsymbol{D}\right). \end{aligned} \tag{9}$$

Note that all of these objects are homogeneous in the momenta although the two Lax matrices are not polynomials. In the following we suppress the index J referring to the Jacobi time and absorb the factor $[2(E-V)]^{-1}$ into the definitions of $\boldsymbol{A}^\alpha$ and $\boldsymbol{D}$, thus focusing on the geodesic Lax pair systems given by

$$\begin{aligned} H &= \tfrac{1}{2}g^{\alpha\beta}p_\alpha p_\beta \\ \boldsymbol{L} &= \boldsymbol{L}^\alpha p_\alpha + \sqrt{2H}\boldsymbol{K} \\ \boldsymbol{A} &= \boldsymbol{A}^\alpha p_\alpha + \sqrt{2H}\boldsymbol{D}. \end{aligned} \tag{10}$$

3. The Lax pair tensors

In [1], a geometrical formulation of the Lax equation was found by demanding that the tensors corresponding to the Lax matrices should be linear and also homogeneous in the momenta. We now generalize this geometrization by taking the system (10) as our starting point. This leads us to write the components of the Lax matrices with mixed indices (i.e. row indices contravariant and column indices covariant) so that $\boldsymbol{L} = (L^\alpha{}_\beta)$, $\boldsymbol{A} = (A^\alpha{}_\beta)$, $\boldsymbol{K} = (K^\alpha{}_\beta)$ and $\boldsymbol{D} = (K^\alpha{}_\beta)$. We then extract the geometrical objects $L^\alpha{}_\beta{}^\gamma$, $A^\alpha{}_\beta{}^\gamma$, from the components of (10)

$$\begin{aligned} L^\alpha{}_\beta &= L^\alpha{}_\beta{}^\gamma p_\gamma + \sqrt{2H}K^\alpha{}_\beta \\ A^\alpha{}_\beta &= A^\alpha{}_\beta{}^\gamma p_\gamma + \sqrt{2H}D^\alpha{}_\beta. \end{aligned} \tag{11}$$

Next, we define connection matrices by $\boldsymbol{\Gamma}^\gamma = (\Gamma^\alpha{}_\beta{}^\gamma)$ where $\Gamma^\alpha{}_\beta{}^\gamma = g^{\gamma\delta}\Gamma^\alpha{}_{\beta\delta}$ and $\Gamma^\alpha{}_{\beta\gamma}$ is the Levi-Civita connection associated to the Jacobi metric. In the following, the Jacobi metric is used to raise and lower indices in this way on other objects as well. Judging from the original Lax tensor formulation we expect that $\boldsymbol{L}^\alpha$, and thereby also $\boldsymbol{K}$, will be tensors. Anticipating this result we replace partial derivatives with covariant derivatives according to the recipes

$$\begin{aligned} \boldsymbol{L}_{\alpha,\beta} &= \boldsymbol{L}_{\alpha;\beta} + [\boldsymbol{L}_\alpha, \boldsymbol{\Gamma}_\beta] - \Gamma_\alpha{}^\gamma{}_\beta \boldsymbol{L}_\gamma \\ \boldsymbol{K}_{,\alpha} &= \boldsymbol{K}_{;\alpha} + [\boldsymbol{K}, \boldsymbol{\Gamma}_\alpha]. \end{aligned} \tag{12}$$

This procedure is justified below. Using (4), the commutator $[\boldsymbol{L}, \boldsymbol{A}]$ of the Lax equation can be written in the form

$$([\boldsymbol{L}^\alpha, \boldsymbol{A}^\beta] + [\boldsymbol{K}, \boldsymbol{D}]g^{\alpha\beta})p_\alpha p_\beta + \sqrt{2H}p_\alpha([\boldsymbol{L}^\alpha, \boldsymbol{D}] + [\boldsymbol{K}, \boldsymbol{A}^\alpha]). \tag{13}$$

The Poisson bracket $\{\boldsymbol{L}, H\}$, on the other hand, using (12), becomes

$$(\boldsymbol{L}^{\alpha;\beta} + [\boldsymbol{L}^{\alpha}, \boldsymbol{\Gamma}^{\beta}])p_\alpha p_\beta + \sqrt{2H}\, p_\alpha(\boldsymbol{K}^{;\alpha} + [\boldsymbol{K}, \boldsymbol{\Gamma}^{\alpha}]). \tag{14}$$

Identifying terms in (13) and (14) yields the following form of the Lax equations:

$$\begin{aligned} \boldsymbol{L}_{(\alpha;\beta)} &= [\boldsymbol{L}_{(\alpha}, \boldsymbol{B}_{\beta)}] + [\boldsymbol{K}, \boldsymbol{D}]g_{\alpha\beta} \\ \boldsymbol{K}_{;\alpha} &= [\boldsymbol{L}_\alpha, \boldsymbol{D}] + [\boldsymbol{K}, \boldsymbol{B}_\alpha] \end{aligned} \tag{15}$$

where we have defined $\boldsymbol{B}^\alpha = \boldsymbol{A}^\alpha - \boldsymbol{\Gamma}^\alpha$. This shows that we can consistently interpret $\boldsymbol{L}$, $\boldsymbol{B}$, $\boldsymbol{K}$ and $\boldsymbol{D}$ as tensors since equations (15) are then manifestly covariant. However, note that $\boldsymbol{A}^\alpha$ is a connection-like object. By setting $\boldsymbol{K} = 0$, $\boldsymbol{D} = 0$, the second of equations (15) becomes an identity while the first equation reduces to the Lax tensor equation of [1].

4. Examples

We now apply the above geometrized Lax formulation to some systems with known Lax representations. They are all of the form (3) with flat kinetic metric $h_{\alpha\beta} = \delta_{\alpha\beta}$. For the systems considered below, it is found that the matrix $\boldsymbol{L}_* := \boldsymbol{L}^\alpha p_\alpha$ is diagonal, while $\boldsymbol{K}$ has no diagonal elements. It follows that $\mathrm{Tr}\,\boldsymbol{K} = 0$ and $\mathrm{Tr}(\boldsymbol{L}_*\boldsymbol{K}) = 0$. In addition, $\boldsymbol{L}_*$ and $\boldsymbol{K}$ satisfy

$$\mathrm{Tr}\,\boldsymbol{L}_* = \sum_\alpha p_\alpha \qquad \mathrm{Tr}(\boldsymbol{L}_*^2) = 2T \qquad \mathrm{Tr}(\boldsymbol{K}^2) = 2V \tag{16}$$

so that the first two invariants of the geometrized system are

$$\begin{aligned} I_1 &= \mathrm{Tr}\,\boldsymbol{L} = \sum_\alpha p_\alpha \\ I_2 &= \tfrac{1}{2}\mathrm{Tr}(\boldsymbol{L}^2) = T + 2HV = \frac{ET}{E-V} = 2EH. \end{aligned} \tag{17}$$

Also, the metric for these models is given by

$$g^{\alpha\beta} = \frac{\mathrm{Tr}(\boldsymbol{L}^\alpha \boldsymbol{L}^\beta)}{2E - \mathrm{Tr}(\boldsymbol{K}^2)} \tag{18}$$

where $\mathrm{Tr}(\boldsymbol{L}^\alpha \boldsymbol{L}^\beta) = h^{\alpha\beta}$. The results obtained below for the different models have been verified for the case $n = 3$, using the package GRTensorII [5] for MapleV.

4.1. The Toda lattice

First, we will consider the n-particle non-periodic (open) Toda lattice, for which

$$V = \sum_{i=1}^{n-1} a_i^2 \qquad a_i = \exp(q^i - q^{i+1}). \tag{19}$$

A standard Lax representation of this system is [6]

$$\begin{aligned} \boldsymbol{L} = L^i{}_j &= \sum_{k=1}^{n} p_k \delta^i{}_k \delta_{jk} + a_k(\delta^i{}_k \delta_{j,k+1} + \delta^i{}_{k+1}\delta_{jk}) \\ \boldsymbol{A} = A^i{}_j &= \sum_{k=1}^{n} a_k(\delta^i{}_k \delta_{j,k+1} - \delta^i{}_{k+1}\delta_{jk}). \end{aligned} \tag{20}$$

From this Lax representation we find

$$\begin{aligned} &\boldsymbol{L}^\alpha = L^i{}_j{}^\alpha = \delta^{i\alpha}\delta_j{}^\alpha \\ &\boldsymbol{K} = K^i{}_j = \sum_{k=1}^{n} a_k(\delta^i{}_k \delta_{j,k+1} + \delta^i{}_{k+1}\delta_{jk}) \\ &\boldsymbol{A}^\alpha = 0 \qquad \boldsymbol{B}^\alpha = -\boldsymbol{\Gamma}^\alpha \qquad \boldsymbol{D} = \frac{1}{2(E-V)}\boldsymbol{A}. \end{aligned} \tag{21}$$

Note that the factor $[2(E-V)]^{-1}$ is absorbed into $\boldsymbol{D}$, as discussed in section 2.

We note in passing that the results for a periodic Toda lattice can be obtained by letting the sum in (19) run from 1 to n and by employing the cyclicity conditions

$$q^{n+1} \to q^1 \qquad \delta^i{}_{n+1} \to \delta^i{}_1 \qquad \delta_{j,n+1} \to \delta_{j1}. \tag{22}$$

4.2. The Calogero–Moser system

Our second example is the Calogero–Moser system (the type I system of [7]). The potential is

$$V = \sum_{i<j} a_{ij}^2 \qquad a_{ij} = \frac{1}{q^i - q^j}. \tag{23}$$

A well known Lax representation of this system is [7]

$$\begin{aligned} \boldsymbol{L} &= L^i{}_j = \sum_{k=1}^{n} p_k \delta^i{}_k \delta_{jk} + \mathrm{i}(1-\delta^i{}_j) a_{ij} \\ \boldsymbol{A} &= A^i{}_j = \mathrm{i}\left(\delta^i{}_j \sum_{k\neq i} a_{ik}^2 - (1-\delta^i{}_j) a_{ij}^2\right) \end{aligned} \tag{24}$$

from which it follows that

$$\begin{aligned} &\boldsymbol{L}^\alpha = L^i{}_j{}^\alpha = \delta^{i\alpha}\delta_j{}^\alpha \\ &\boldsymbol{K} = K^i{}_j = i(1-\delta^i{}_j) a_{ij} \\ &\boldsymbol{A}^\alpha = 0 \qquad \boldsymbol{B}^\alpha = -\boldsymbol{\Gamma}^\alpha \qquad \boldsymbol{D} = \frac{1}{2(E-V)}\boldsymbol{A}. \end{aligned} \tag{25}$$

5. Four-dimensional spacetime generalizations

In general relativity, symmetries of spacetime play an important role. Many examples of spacetimes with linear invariants, corresponding to Killing vectors, are known [8]. However, higher-order invariants (apart from the trivial quadratic invariant $g^{\alpha\beta} p_\alpha p_\beta$, associated with the metric) are quite rare. A well known example is the second-rank Killing tensor of the Kerr spacetime [9], which together with the two existing Killing vectors enable a complete integration of the geodesics of that system. Examples of spacetimes with a nontrivial third rank Killing tensor were given in [2], but apart from that, such Killing tensors are, to our knowledge, unknown. In this perspective, it is of interest to extend the Riemannian geometries obtained above (with $n = 3$) to four-dimensional spacetimes that inherit the symmetries of the original geometry. The simplest generalization is obtained by introducing a time-like coordinate q^0 according to

$${}^{(4)}\mathrm{d}s^2 = -(\mathrm{d}q^0)^2 + \mathrm{d}s^2 \tag{26}$$

where $\mathrm{d}s^2 = 2(E-V)h_{\alpha\beta}\,\mathrm{d}q^\alpha\,\mathrm{d}q^\beta$, $\alpha, \beta \in \{1, 2, 3\}$, and V is one of the potentials studied above. Note that the condition $E > V$ must be satisfied for the metric to have a Lorentzian signature. This is not a serious drawback, since the sign of $E - V$ is preserved along the geodesics of the metric (26). Hence, a completely integrable $(1 + 3)$-dimensional geometry is well defined in the region $E > V$. In general, these spacetimes are of Petrov type II. Choosing a particular potential V, the Petrov type may be further specialized. The Toda and Calogero–Moser potentials both lead to Petrov type II spacetimes. In a Lorentzian frame corresponding to the given coordinates, the energy–momentum tensor T^{ab}, $a, b \in \{0, 1, 2, 3\}$, for a spacetime of this type satisfies $T^{0\alpha} = 0$ and can thus be diagonalized using a rotation of the spatial part of the frame. Another characteristic feature is the fact that the eigenvalue equation for the spatial part of T^{ab} naturally factorizes into one linear and one second-degree equation. This is due to the existence of the space-like Killing vector Tr $\boldsymbol{L}^\alpha$. The linear equation gives the anisotropic pressure in the direction of this Killing vector. In terms of $\tilde{T}^{ab} = 2(E-V)^3 T^{ab}$, the Lorentzian frame components of the energy–momentum tensor for the Toda spacetime become

$$\begin{aligned}
\tilde{T}^{00} &= 8EV - 2\sum_{i=1}^{3} a_i^2(a_i^2 + 11a_{i+1}^2) \\
\tilde{T}^{\alpha\alpha} &= -2(V + a_{\alpha+1}^2)E + (V + 9a_{\alpha+1}^2)(V - a_{\alpha+1}^2) \\
\tilde{T}^{\alpha,\alpha+1} &= (-2E + 5V - 6a_\alpha^2)a_\alpha^2 - 3a_{\alpha-1}^2 a_{\alpha+1}^2
\end{aligned} \tag{27}$$

where it is to be understood that indices are added modulo three. These expressions hold for the open case ($a_3 = 0$) as well as for the closed case ($a_1a_2a_3 = 1$). Diagonalizing the energy–momentum tensor gives $\tilde{T}^{ab} = \mathrm{diag}\,(\tilde{\mu}, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$, where

$$\begin{aligned}
\tilde{\mu} &= \tilde{T}^{00} \\
\tilde{p}_1 &= -4EV + 12\sum_{i=1}^{3} a_i^2 a_{i+1}^2 \\
\tilde{p}_{2,3} &= \Lambda \pm \sqrt{\Lambda^2 - \Delta} \\
\Lambda &= -2EV + \sum_{i=1}^{3} a_i^2(a_i^2 + 5a_{i+1}^2) \\
\Delta &= 12(-13E + 9V)a_1^2a_2^2a_3^2 \\
&\qquad + 2\sum_{i=1}^{3} a_i^2\{6E^2a_{i+1}^2 - E[4a_i^4 + 3a_{i+1}^2(a_i^2 + a_{i+1}^2)] + 9a_{i+1}^2(a_i^2 - a_{i+1}^2)^2\}.
\end{aligned} \tag{28}$$

Since the eigenvalues of T^{ab} are related to those of $\tilde{T}^{ab}$ by a positive factor whenever the metric signature condition $E > V$ holds, the weak energy condition [10] reads $\tilde{\mu} \geqslant 0$, $\tilde{\mu} + \tilde{p}_\alpha \geqslant 0$ which is equivalent to $\tilde{\mu} \geqslant 0$, $\tilde{\mu} + \tilde{p}_1 \geqslant 0$, $\tilde{\mu} + \Lambda \geqslant 0$, $\Delta \geqslant 0$. At the spatial origin $q^\alpha = 0$, where $V = 2$ ($V = 3$) for the open (closed) case, these inequalities are satisfied if $E > 2$ ($E > 3$). Hence by continuity, there must be some open region in (E, q^α)-space where the metric signature condition and the weak energy condition hold simultaneously.

Calculating T^{ab} for the Calogero–Moser spacetime yields the result

$$\begin{aligned}
\tilde{T}^{00} &= 3\sum_{i=1}^{3} 4Ea_i^4 - (a_i^3 + a_{i+1}^3)^2 - 4a_i^2a_{i+1}^2(a_i^2 + a_{i+1}^2) \\
\tilde{T}^{\alpha\alpha} &= -3E(a_{\alpha-1}^4 + a_\alpha^4 + 2a_{\alpha+1}^4) + 3(a_{\alpha-1}^4 + a_\alpha^4)a_{\alpha+1}^2 + 3(V - a_{\alpha+1}^2)(a_{\alpha-1}^2a_\alpha^2 + 2a_{\alpha+1}^4) \\
&\qquad -2a_{\alpha-1}^3a_\alpha^3 + 4(a_{\alpha-1}^3 + a_\alpha^3)a_{\alpha+1}^3 + 2\sum_{i=1}^{3} a_i^6 \\
\tilde{T}^{\alpha,\alpha+1} &= -3a_\alpha^4(E - V + a_\alpha^2) + 3a_\alpha^3(a_{\alpha-1}^3 + a_{\alpha+1}^3) - 3a_{\alpha-1}^3a_{\alpha+1}^3
\end{aligned} \tag{29}$$

where $\tilde{T}^{ab}$ is defined as in the Toda case. As a Calogero–Moser system with three particles can be viewed as a system with nearest-neighbour interaction, we have used the notation $a_i = (q^i - q^{i+1})^{-1}$, $V = \sum_{i=1}^3 a_i^2$. The eigenvalues of $\tilde{T}^{ab}$ can be written down in a form analogous to the Toda case. We choose not to do so here, however, as the expressions are more complicated and not very illustrative. As for the Toda system, there is a region where the weak energy condition is satisfied. This can be verified, e.g. by setting $q^1 = -1$, $q^2 = 0$, $q^3 = 1$, which gives $V = \frac{9}{4}$ and the following eigenvalues for $\tilde{T}^{ab}$:

$$\begin{aligned} \tilde{\mu} &= \tfrac{99}{4}(E - \tfrac{545}{264}) \\ \tilde{p}_1 &= -\tfrac{99}{8}(E - \tfrac{793}{396}) \\ \tilde{p}_{2,3} &= -\tfrac{99}{16}(E - \tfrac{421}{198}) \pm \tfrac{3}{16}|15E - 46|. \end{aligned} \tag{30}$$

For $E > \frac{9}{4}$, these eigenvalues satisfy the weak energy condition.

6. Discussion

In this paper we have presented a general procedure for obtaining a tensorial Lax representation from known Lax pair matrices. The tensorial representation is obtained via a time reparametrization. It has two dynamical Lax tensors ($\boldsymbol{L}$ and $\boldsymbol{K}$), instead of only $\boldsymbol{L}$ as in the original geometric formulation [1]. One advantage with this particular geometrical formulation is that it provides a straightforward recipe for obtaining Lax tensors from a known Lax representation. Indeed, any n-dimensional Hamiltonian system of the classical form (3) that has a Lax representation of dimension n can be geometrized in this way.

References

[1] Rosquist K 1997 *On Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories* vol 1, ed R T Jantzen and G M Keiser (Singapore: World Scientific) p 379

[2] Rosquist K and Goliath M 1998 *Gen. Rel. Grav.* **30** 1521

[3] Lanczos C 1986 *The Variational Principles of Mechanics* (New York: Dover)

[4] Hietarinta J, Grammaticos B and Ramani A 1984 *Phys. Rev. Lett.* **53** 1707

[5] Musgrave P, Pollney D and Lake K 1996 *GRTensorII, booklet A-F* `http://www.astro.queensu.ca/~grtensor/GRHome.html`

[6] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* vol I (Basel: Birkhäuser)

[7] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313

[8] Kramer D, Stefani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Berlin: VEB Deutscher Verlag der Wissenschaften)

[9] Walker M and Penrose R 1970 *Commun. Math. Phys.* **18** 265

[10] Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press)